A Closed Prime Number Test Function

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Abstract

This paper will introduce the Multiple Test function, which will then be reduced to the Integer Test function. Then, we will introduce its counterpart, the Non-Integer Test Function. Finally, using the Integer Test function and Non-Integer Test function, we will derive both an algorithm and a closed Prime Number Test function that computes whether or not a certain number is prime.

Keywords

integer, test, function, algorithm, closed, prime, number

Introduction

Since the discovery of prime numbers, there have been several questions concerning them, one of which is whether or not a particular natural number is prime or composite, as there is no apparent pattern for their emergence.

A prime number, by definition, is a natural number that has only two divisors, namely 1 and itself[1]. A natural number is composite otherwise. However, the number 1 is neither prime or composite since, even though it is divisible by 1 and itself, it only has one divisor. Composite numbers have more than two divisors.

Multiple Test, Integer Test, and Non-Integer Test Functions

To begin, let us derive the Multiple Test function, which allows us to determine whether or not p is divisible by q. It is well-known in binary that a bit equal to 1 indicates yes and a bit equal to 0 indicates no, so we will adopt this convention[2]. In essence, we are looking for a function that computes:

$$f(p,q) = \begin{cases} 0 & q \nmid p \\ 1 & q \mid p \end{cases} \tag{1}$$

As we can see, this is a piecewise function. Let us construct a closed formula that is equivalent. Since (1) is bounded by $[0,1]$ and we know that multiples are periodic, we can construct such a function using an oscillating function, namely cosine. By observation, the period is $\frac{q}{2\pi}$. To match the bounds, we set the amplitude and vertical shift to $\frac{1}{2}$:

$$\tilde{M}(p,q) = \frac{1}{2}\left(\cos\left(\frac{2\pi p}{q}\right) + 1\right) \tag{2}$$

This evaluates to 1 when p is divisible by q and some value $v \in [0,1)$ when p is not divisible by q. We can now implement the floor function[3]. Since $v \in [0,1)$ when $q \nmid p$, then $\lfloor v \rfloor = 0$. Furthermore, when $q \mid p$, $\lfloor 1 \rfloor = 1$. Therefore, we place (2) inside the floor function to obtain a closed formula that is equivalent to (1):

$$M(p,q) = \left\lfloor \frac{1}{2}\left(\cos\left(\frac{2\pi p}{q}\right) + 1\right) \right\rfloor \tag{3}$$

We can even construct a more compact form of this by using the sum identity for cosine[4]:

$$\begin{aligned}
\cos\left(\frac{\pi p}{q} + \frac{\pi p}{q}\right) &= \cos\left(\frac{\pi p}{q}\right)\cos\left(\frac{\pi p}{q}\right) - \sin\left(\frac{\pi p}{q}\right)\sin\left(\frac{\pi p}{q}\right) \\
\to \cos\left(\frac{2\pi p}{q}\right) &= \cos^2\left(\frac{\pi p}{q}\right) - \sin^2\left(\frac{\pi p}{q}\right) \\
&= \cos^2\left(\frac{\pi p}{q}\right) - \left(1 - \cos^2\left(\frac{\pi p}{q}\right)\right) \\
&= 2\cos^2\left(\frac{\pi p}{q}\right) - 1 \\
\to \cos\left(\frac{2\pi p}{q}\right) + 1 &= 2\cos^2\left(\frac{\pi p}{q}\right) \\
\to \frac{1}{2}\left(\cos\left(\frac{2\pi p}{q}\right) + 1\right) &= \cos^2\left(\frac{\pi p}{q}\right)
\end{aligned} \tag{4}$$

The LHS is exactly the expression inside the floor function in (3). So, by substitution, we have a more compact formula for the Multiple Test function:

$$M\left(p,q\right)=\left\lfloor\cos^2\left(\frac{\pi p}{q}\right)\right\rfloor \tag{5}$$

If we set $q=1$, this computes 1 when p is divisible by 1 and otherwise 0, which is the same as stating when p is an integer. Thus, we have the Integer Test function:

$$I\left(p\right)=\left\lfloor\cos^2\left(\pi p\right)\right\rfloor \tag{6}$$

Now, we construct the Integer Test function's counterpart, the Non-Integer Test function, which has the following conditions:

$$f\left(p\right)=\begin{cases} 0 & p\in\mathbb{Z} \\ 1 & p\notin\mathbb{Z}\end{cases} \tag{7}$$

We could simply subtract (6) from 1:

$$I^{-1}\left(p\right)=1-\left\lfloor\cos^2\left(\pi p\right)\right\rfloor \tag{8}$$

But let us be creative and place the 1 inside of the floor function:

$$\tilde{I}^{-1}\left(p\right)=\left\lfloor 1-\cos^2\left(\pi p\right)\right\rfloor=\left\lfloor\sin^2\left(\pi p\right)\right\rfloor \tag{9}$$

Unfortunately, (8) and (9) are not equivalent. In (8), the difference is between two natural numbers, but in (9), the difference is between a natural number and a real number. By construction of the Integer Test function, we know that $\left\lfloor\cos^2\left(\pi p\right)\right\rfloor=1$ when p is an integer. Hence, $1-\left\lfloor\cos^2\left(\pi p\right)\right\rfloor=0$. This agrees with the first condition for the Non-Integer Test function. To force the second condition, we note that when p is not an integer, $\cos^2\left(\pi p\right)\in[0,1)$, which implies $\sin^2\left(\pi p\right)\in(0,1]$. Now, instead of using the floor function, we implement the ceiling function[3]. Since $\sin^2\left(\pi p\right)\in(0,1]$ when p is not an integer, then $\left\lceil\sin^2\left(\pi p\right)\right\rceil=1$. Furthermore, when p is an integer, $\lceil 0\rceil=0$. Therefore, we place $\sin^2\left(\pi p\right)$ inside the ceiling function to obtain a closed formula that is equivalent to (7) and (8):

$$I^{-1}\left(p\right)=\left\lceil\sin^2\left(\pi p\right)\right\rceil \tag{10}$$

In the next section, we will discover how both the Integer Test function and Non-Integer Test function are important in deriving two separate functions, one re-iterative and one closed, for testing when a natural number is prime.

Prime Number Test Functions

From our definition of a prime number stated in the introduction, a prime number is solely based on the number of divisors it has. A divisor d_i of a natural number n, by definition, must be a natural number between 1 and n. Intuitively, we can express any divisor of n in the following way:

$$\tilde{d}_i(n) = \frac{n}{i} \tag{11}$$

where $1 \leq i \leq n$. But $\tilde{d}_i$ may not always be a natural number. Therefore, we must use the Integer Test Function:

$$I\left(\tilde{d}_i(n)\right) = \left\lfloor \cos^2\left(\frac{\pi n}{i}\right)\right\rfloor \tag{12}$$

This computes either a 0 or 1 for each i, depending on if $\frac{n}{i}$ is an integer. So, if (12) evaluates to 1, then a divisor d_i of n can be computed by the following:

$$d_i(n) = \frac{n}{i}\left\lfloor \cos^2\left(\frac{\pi n}{i}\right)\right\rfloor \tag{13}$$

If we sum the values over all i's in (12), then we obtain a version of the Integer Counting function:

$$C\left(\tilde{d}_i(n)\right) = \sum_{i=1}^{n}\left\lfloor \cos^2\left(\frac{\pi n}{i}\right)\right\rfloor \tag{14}$$

which computes the number of divisors of n.

Clearly, whether or not n is prime, n is always divisible by 1 and n. So, let us put aside these two divisors, and we do so by subtracting 2 from (14):

$$\sum_{i=1}^{n}\left\lfloor \cos^2\left(\frac{\pi n}{i}\right)\right\rfloor - 2 \tag{15}$$

This computes the number of non-trivial divisors of n. Note that this holds true only if $n > 1$, for the reason stated in the introduction.

Obviously, i can never exceed n. Hence, $\frac{C\left(\tilde{d}_i(n)\right)}{n} \leq 1$. Moreover, $0 \leq \frac{C\left(\tilde{d}_i(n)\right)-2}{n} < 1$. Consider dividing (15) by n:

$$\frac{1}{n}\left(\sum_{i=1}^{n}\left\lfloor \cos^2\left(\frac{\pi n}{i}\right)\right\rfloor - 2\right) \tag{16}$$

This evaluates to 0 if there are no non-trivial divisors and some value $v \in (0,1)$ if there are non-trivial divisors. Suppose we subtract this from 1:

$$\frac{n-1}{n}\left(\sum_{i=1}^{n}\left\lfloor\cos^2\left(\frac{\pi n}{i}\right)\right\rfloor-2\right) \tag{17}$$

This evaluates to 1 if there are no non-trivial divisors and some value $v \in (0,1)$ if there are non-trivial divisors. If there are non-trivial divisors, we can force v to equal 0 by placing (17) inside the floor function:

$$P\left(n\right)=\left\lfloor\frac{n-1}{n}\left(\sum_{i=1}^{n}\left\lfloor\cos^2\left(\frac{\pi n}{i}\right)\right\rfloor-2\right)\right\rfloor \tag{18}$$

This is the re-iterative algorithm for the Prime Number Test function.

Now, we construct a closed formula for the Prime Number Test function. Consider the following rational number:

$$\frac{(n-1)!}{n} \tag{19}$$

If n is prime, then no term in the factorial is divisible by n, as each term is less than n and the only numbers divisible by n are integer multiples of n. Hence, (19) is not an integer when n is prime.

We need to check whether or not (19) is an integer if n is composite. It is easy to show that multiples grow faster than powers. Let $\{p_k\}$ be a sequence of distinct primes that are between 2 and n, in increasing order. Then:

$$\left\lfloor\frac{n}{p_k}\right\rfloor \geq \left\lfloor\log_{p_k} n\right\rfloor \tag{20}$$

Since the numerator in (19) is a factorial, it is a product of terms j, $1 \leq j \leq n-1$. So, we now look at what happens when we place $n-1$ in the numerator on the LHS. If $p_k \not| n$ and we subtract 1 from the numerator on the LHS, then the value on the LHS in (20) does not change. On the other hand, if $p_k|n$ and we subtract 1 from the numerator on the LHS, then we subtract one multiple (i.e. $\left\lfloor\frac{n-1}{p_k}\right\rfloor = \left\lfloor\frac{n}{p_k}\right\rfloor - 1$). We then show that the following inequality holds:

$$\left\lfloor\frac{n}{p_k}\right\rfloor - 1 \geq \left\lfloor\log_{p_k} n\right\rfloor \tag{21}$$

Rearranging this, we have:

$$\left\lfloor\frac{n}{p_k}\right\rfloor - \left\lfloor\log_{p_k} n\right\rfloor \geq 1 \tag{22}$$

By (20), this holds true, except for two cases, observing the equality. The first case is when $n = p_k$, as $\left\lfloor \frac{n}{p_k} \right\rfloor = 1 = \lfloor \log_{p_k} n \rfloor$. But we already stated that n is considered to be composite throughout this reasoning. The second case is when $n = 4$ and $p_k = 2$, as $\left\lfloor \frac{n}{p_k} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor = \lfloor 2 \rfloor = 2 = \lfloor \log_{p_k} n \rfloor = \lfloor \log_2 4 \rfloor = \lfloor 2 \rfloor = 2$. Note 4 is composite. For both cases, we arrive at $0 \geq 1$, which is certainly not true. For now, let us put aside the case when $n = 4$ and suppose (22) is true for all $n \in \mathbb{N}^+$. We will re-visit this case shortly. Since (22) is true, this implies there are terms in the factorial in (19) that are divisible by n, allowing (19) to be an integer when n is composite.

We now have that (19) is an integer when n is composite and not an integer when n is prime. Therefore, by the use of the Non-Integer Test function, we can express the closed formula of the Prime Number Test function as:

$$\tilde{P}(n) = \left\lceil \sin^2 \left(\frac{\pi (n-1)!}{n} \right) \right\rceil \tag{23}$$

Returning to the case when $n = 4$, (19) can be expressed in the following manner:

$$\frac{1 \cdot 2 \cdot 3}{4} = \frac{1 \cdot 2 \cdot 3}{2 \cdot 2} = \frac{1 \cdot 3}{2} = \frac{3}{2} \tag{24}$$

Clearly, $\frac{3}{2}$ is not an integer. But we can force it to be an integer by multiplying the numerator by 2. By doing so, we modify (23) to:

$$\hat{P}(n) = \left\lceil \sin^2 \left(\frac{2\pi (n-1)!}{n} \right) \right\rceil \tag{25}$$

But this fails when $n = 2$ since $\hat{P}(2) = 0$, but we know 2 is prime. The reason this fails is because $2 \cdot (n-1)! = 2 \cdot 1! = 2 \cdot 1 = 2 = n$. The numerator in (19) needs to be a product of terms j, $1 \leq j \leq n-1$. So, we need to express 2 in terms of n such that the expression produces some value equal to one of the terms in the factorial. We can do so by $2 = \sqrt{n}$, since $n = 4$. But for $n = 2$, $\sqrt{2}$ is not an integer, nor is it a term in the factorial. To force it to be an integer, we once again place it inside the floor function. Now, $\lfloor \sqrt{2} \rfloor = 1$, which is a term in the factorial when $n = 2$. From the sum of consecutive odd numbers[5], we have $n - \lfloor \sqrt{n} \rfloor \geq 1$, or $\lfloor \sqrt{n} \rfloor \leq n - 1$. Furthermore, when n is prime, we know that no terms in the factorial are divisible by n. So, neither is $\lfloor \sqrt{n} \rfloor$. Likewise, if n is composite, then $\lfloor \sqrt{n} \rfloor$ is an extra term in the product in the numerator of (19), which still produces an integer, as desired.

In conclusion, the correct closed formula for the Prime Number Test function that holds true for all $n \in \mathbb{N}^+$ is:

$$P(n) = \left\lceil \sin^2 \left(\frac{\pi (n-1)! \lfloor \sqrt{n} \rfloor}{n} \right) \right\rceil \tag{26}$$

Future

There are two endeavors that I wish to embark based on this paper. The first is to determine a closed formula for the Integer Counting function, possibly by equating (18) and (26). The second is to determine a closed formula for the n^{th} prime number, using the Prime Number Test function.

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