A New Closed Formula for Doubly Triangular Numbers

Ryan Jean
March 28, 2021

Abstract

This paper will propose and confirm a new closed formula for the doubly triangular numbers.

Keywords

doubly, triangular, number, color, square, edge, vertex, symmetry, rotation, reflection

Introduction

The doubly triangular numbers describe the number of inequivalent ways to color the edges (or vertices) of a square using k colors, allowing rotations and reflections. We will prove the claim that these numbers can be calculated by using the new closed formula:

$$T\left(k\right) = 2k^2 - k + 6\binom{k}{3} + 3\binom{k}{4} \tag{1}$$

Proof

In this proof, we will consider edges, but vertices work in the same manner due to their symmetry with edges (in the case of a square). To begin, let us consider each color k as a separate entity and build from there. We obviously know that $k \in \mathbb{N}$. In the simplest case where $k = 0$, there are no colors to choose from and thus, $T(0) = 0$. The next simplest case is $k = 1$. With only one color, we can color all the edges of a square only one color. Furthermore, for any k, there is only one way to color all the edges the same color. But because there are k colors to choose from, we must sum all these ones up to k. In other words, suppose $A(k)$ is the number of ways to color all four edges of a square the same color with k colors, then:

$$A(k) = \sum_{i=1}^{k} 1 = k \tag{2}$$

Now, let us consider the number of inequivalent ways to color edges of a square with only two colors, allowing rotations and reflections. There are four: three edges with color blue and one edge with red, three edges with color red and one edge with blue, and two ways with two edges of each color. The last reflects the case where two colors are adjacent and the case where the two colors are alternating. Suppose we fix the color red, which is our k^{th} color. Then, there are $k-1$ colors we could have for our opposing color. As we did before, we need to sum all of these up to $k-1$ as any color can be fixed, not just the k^{th} color. If we allow $B(k)$ to be the number of inequivalent ways to color edges of a square with choices of two colors out of k, then:

$$B(k) = \sum_{i=1}^{k-1} 4i = 4\sum_{i=1}^{k-1} i = 4\left(\frac{k(k-1)}{2}\right) = 4\left(\frac{k^2-k}{2}\right) = 2\left(k^2-k\right) = 2k^2 - 2k \tag{3}$$

We now consider cases where $k > 2$, but we are choosing three colors out of k to color the four edges of a square. Let us fix color k so that it colors only one edge. This means we have $k-1$ colors to choose from to color three edges. However, we want those edges to be two different colors. There are $\binom{k-1}{2}$ ways to do this. The two colors can also be reversed. Therefore, this is doubled to $2\binom{k-1}{2}$. There are also two ways of arranging the two colors (adjacent and alternating), as stated in the previous paragraph. Thus, the value doubles again to $4\binom{k-1}{2}$. We must sum all of these up to $k-1$ as any color can be fixed. Suppose $C(k)$ is the number of inequivalent ways to color edges of a square with choices of at least three colors where one color colors only one edge. Then:

$$C(k) = \sum_{i=3}^{k-1} 4\binom{i-1}{2} = 4\sum_{i=3}^{k-1} \binom{i-1}{2} = 4\binom{k}{3} \tag{4}$$

Suppose now that our k^{th} color is still fixed, but instead colors two of the four edges of a square. We need the two remaining edges to be two different colors. Since our k^{th} color is fixed, we have $k-1$ colors to choose from, which yields $\binom{k-1}{2}$. Once again, the two colors can be reversed, which means this value doubles to $2\binom{k-1}{2}$. Again, we need to sum all of these up to $k-1$ as any color can be fixed. If we allow $D(k)$ to be the number of inequivalent ways to color edges of a square with choices of at least three colors where one color colors two edges, then:

$$D(k) = \sum_{i=3}^{k-1} 2\binom{i-1}{2} = 2\sum_{i=3}^{k-1}\binom{i-1}{2} = 2\binom{k}{3} \tag{5}$$

There are no other ways to color the edges of a square using three different colors.

We now consider the case where we color each edge of a square a different color. As we did before, our k^{th} color is fixed and colors only one edge. Therefore, there are $k-1$ colors to choose from to color the remaining three edges, allowing $\binom{k-1}{3}$ ways to do so. The two sets of adjacent colors can be reversed and the two parallel colors can be reversed, tripling the value to $3\binom{k-1}{3}$. Once more, we need to sum all of these up to $k-1$ as any color can be fixed. Suppose $E(k)$ is the number of inequivalent ways to color edges of a square a different color. Then:

$$E(k) = \sum_{i=4}^{k-1} 3\binom{i-1}{3} = 3\sum_{i=4}^{k-1}\binom{i-1}{3} = 3\binom{k}{4} \tag{6}$$

Our function $T(k)$ is simply a sum of all of the functions we just derived, as it is the number of inequivalent ways to color the edges of a square using k colors, allowing rotations and reflections. Therefore:

$$\begin{aligned} T(k) &= A(k) + B(k) + C(k) + D(k) + E(k) \\ &= k + 2k^2 - 2k + 4\binom{k}{3} + 2\binom{k}{3} + 3\binom{k}{4} = 2k^2 - k + 6\binom{k}{3} + 3\binom{k}{4} \end{aligned} \tag{7}$$

Since this formula matches formula (1), our claim has been proven. We have a new closed formula for the doubly triangular numbers, which is formula (1)/(7).