An Explicit Algorithm for the Exact Number of Integer Lattice Points in an N-Ball

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Abstract

This paper will introduce the Multiple Test function, which will then be reduced to the Integer Test function. Using the Integer Test function, an explicit algorithm will be derived for an exact solution to the Gauss Circle Problem. This result will then be generalized to provide an algorithm for any n-ball.

Keywords

Gauss, circle, integer, lattice, n-ball, n-sphere, multiple

Introduction

In Euclidean space, an n-ball, denoted $B(\vec{a}, r)$, is defined as the set of all points in $\mathbb{R}^n$ within a distance (or radius) r, the magnitude of an arbitrary vector $\vec{v}$ in $\mathbb{R}^n$, from its center point $\vec{a}$ in $\mathbb{R}^n$[1]. If the n-ball is open, it does not contain its boundary points, called an n-sphere. If it is closed, it does contain its boundary points[2]:

$$B(\vec{a}, r) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| \leq r\} \tag{1}$$

Although its origin is undetermined, it was Carl Friedrich Gauss, a German mathematician (1777-1855), who was the first to make progress on what is now known as the Gauss Circle Problem[3].

The Gauss Circle Problem is the problem of determining how many integer lattice points lie in a closed disk (2-ball) with a certain radius centered at the origin. Gauss was able to approximate the number using the area of the circle. The approximation is:

$$N(r) = \pi r^2 + E(r) \tag{2}$$

where $E(r)$ is the error and:

$$|E(r)| \leq 2\pi r\sqrt{2} \tag{3}$$

Others have found exact implicit solutions to the problem (one of which has a bound at infinity and another in terms of $r_2(i)$, the Sum of Squares function, the number of ways of writing i as the sum of two squares):

$$N(r) = 1 + 4\sum_{i=0}^{\infty}\left(\left\lfloor\frac{r^2}{4i+1}\right\rfloor - \left\lfloor\frac{r^2}{4i+3}\right\rfloor\right) = \sum_{i=0}^{r^2} r_2(i) \tag{4}$$

Multiple Test and Integer Test Functions

To begin, let us derive the Multiple Test function, which allows us to determine whether or not p in $\mathbb{R}$ is divisible by q in $\mathbb{R}$. It is well-known in binary that a bit equal to 1 indicates yes and a bit equal to 0 indicates no, so we will adopt this convention[4]. In essence, we are looking for a function that computes:

$$f\left(p,q\right)=\left\{\begin{array}{ll} 0 & q\not|p \\ 1 & q|p \end{array}\right. \tag{5}$$

This is a piecewise function, which can be unpleasant at times. Using the floor function, we can transform this into a function that is not piecewise. Recall that the floor function can be written as[5]:

$$\left\lfloor\frac{p}{q}\right\rfloor=\frac{p}{q}-\left\{\frac{p}{q}\right\} \tag{6}$$

where the curly braces indicate the decimal part of the fraction inside. These curly braces are the crucial part of our function. If p is divisible by q, then $\left\{\frac{p}{q}\right\}=0$. If not, then $0<\left\{\frac{p}{q}\right\}<1$. The first half is the opposite of what we want. However, if we subtract the curly braces from 1, the first half computes what we want. In other words, if p is divisible by q, then $1-\left\{\frac{p}{q}\right\}=1$. If not, then $0<1-\left\{\frac{p}{q}\right\}<1$. To allow the second half to become 0, all we need to do is take the floor of $1-\left\{\frac{p}{q}\right\}$ since it can never equal 1. Thus, we now have a function that is not piecewise to replace equation (5), which is one version of the Multiple Test function:

$$M\left(p,q\right)=\left\lfloor 1-\left\{\frac{p}{q}\right\}\right\rfloor \tag{7}$$

The curly braces are the sawtooth function and can be written as[6]:

$$\left\{\frac{p}{q}\right\}=\frac{1}{2}-\frac{1}{\pi}\tan^{-1}\left(\cot\left(\frac{p\pi}{q}\right)\right) \tag{8}$$

Therefore, another way of writing the Multiple Test function is:

$$M\left(p,q\right)=\left\lfloor\frac{1}{\pi}\tan^{-1}\left(\cot\left(\frac{p\pi}{q}\right)\right)+\frac{1}{2}\right\rfloor \tag{9}$$

If we analyze the Multiple Test function, we can see that it oscillates between 0 and 1 with a period of q. Using this information, we can conclude that the function is sinusoidal. Its amplitude and vertical shift are both equal to $\frac{1}{2}$. We only need to include the floor function so that we only obtain values of 0 and 1. Thus, our final version (and the one we will use for the remainder of this paper) of the Multiple Test function is:

$$M(p,q) = \left\lfloor \frac{1}{2}\left(\cos\left(\frac{2\pi p}{q}\right)+1\right)\right\rfloor \tag{10}$$

If we set $q = 1$, this computes 1 whenever p divides 1 and otherwise 0, which is the same as stating whenever p is an integer. Thus, we have the Integer Test function:

$$I(p) = \left\lfloor \frac{1}{2}\left(\cos\left(2\pi p\right)+1\right)\right\rfloor \tag{11}$$

These two equations (10) and (11) can come in handy when we wish to have certain multiples or integers abide by an additional function. For instance, suppose we have the following equation:

$$y = x^2 \tag{12}$$

Assuming we are in $\mathbb{R}^2$, every value of x is squared. Let us change it to:

$$y = M(x,q)\,x^2 \tag{13}$$

Only the values of x that are multiples of q are squared and the function is 0 elsewhere. Now, let us change it to:

$$y = I(x)\,x^2 \tag{14}$$

In this case, only when $x \in \mathbb{Z}$ is it squared and the function is 0 elsewhere.

In the next section, we will discover how the Integer Test function is important in determining an exact explicit solution to the Gauss Circle Problem and how we can extend this knowledge to find an exact explicit solution to any n-ball.

Gauss Circle Problem and N-Balls

The Gauss Circle Problem deals with a closed disk, so let us first concentrate on the boundary of the closed disk, which is a circle. For the remainder of this paper, we will also use the convention in the Gauss Circle Problem and assume all n-balls are centered at the origin (i.e. $\|\vec{a}\| = 0$). The general equation for a circle is:

$$x^2 + y^2 = r^2 \tag{15}$$

We can apply the Integer Test function to this to obtain:

$$I(x)\, x^2 + I(y)\, y^2 = r^2 \tag{16}$$

This only evaluates when both coordinates, x and y, are integers and their mapped point lies on the circle with radius r.

Now, we wish to calculate how many points with these properties there are. Each set of coordinates has three counterparts that are also integers. The signs of the coordinates are (++, -+, –, +-). Therefore, we can divide the circle into quadrants. Also, for our purposes, we are assuming each half-axis is a part of each quadrant. For simplicity, let us work in Quadrant I. We can rearrange equation (16) to have y on the left and in terms of x:

$$y\sqrt{I(y)} = \sqrt{r^2 - I(x)\, x^2} \tag{17}$$

We know that the Integer Test function only computes 0 or 1. The square root of 0 is 0 and the square root of 1 is 1. Thus:

$$\sqrt{I(y)} = I(y) \tag{18}$$

We can substitute this into equation (17) to obtain:

$$I(y)\, y = \sqrt{r^2 - I(x)\, x^2} \tag{19}$$

The $I(y)$ only tests whether or not y is an integer, so we can exclude this from the left and obtain an equation for y:

$$y = \sqrt{r^2 - I(x)\, x^2} \tag{20}$$

We now obtain an equation for $I(y)$:

$$I(y) = I\left(\sqrt{r^2 - I(x)\, x^2}\right) \tag{21}$$

We take note that x must also be an integer, and thus, both coordinates must be integers. Since this computes either a 0 or 1, it also acts as a counting function for the number of integer lattice points that lie on the circle with radius r. To calculate the number, we simply include a summation:

$$\sum I\left(\sqrt{r^2 - I(x)\,x^2}\right) \tag{22}$$

Since we are summing over the x's discretely, we automatically know that x is considered an integer, so we can exclude $I(x)$:

$$\sum I\left(\sqrt{r^2 - x^2}\right) \tag{23}$$

Suppose $\vec{v}$ lies on the x-axis with its tail at the origin and pointed in the positive direction (per our definition of Quadrant I). The possible values of x are $0 \leq x \leq r$. Due to the discrete summation, we must take the floor of r for the upper bound. Now, we have our bounds:

$$\sum_{i=0}^{\lfloor r \rfloor} I\left(\sqrt{r^2 - i^2}\right) \tag{24}$$

However, if we apply these bounds to x, we would obtain two sets of coordinates: $(0, r)$ and $(r, 0)$. Although these are both valid possibilities, the latter point lies on a different quadrant. Thus, we are double counting per half-axis. To prevent this, we can subtract the point at the head of $\vec{v}$ if it is an integer. We can do so by applying the Integer Test:

$$\sum_{i=0}^{\lfloor r \rfloor} \left(I\left(\sqrt{r^2 - i^2}\right)\right) - I(r) \tag{25}$$

Since there are four quadrants, we multiply this by 4 to obtain the number of integer lattice points on the circle:

$$P(2, r) = 4\left(\sum_{i=0}^{\lfloor r \rfloor} \left(I\left(\sqrt{r^2 - i^2}\right)\right) - I(r)\right) \tag{26}$$

This is a function of n and r, and $n = 2$ in this case. Let us separate the first term from the case when $i = 0$:

$$\sum_{i=0}^{\lfloor r \rfloor} \left(I\left(\sqrt{r^2 - i^2}\right)\right) = I(r) + \sum_{i=1}^{\lfloor r \rfloor} \left(I\left(\sqrt{r^2 - i^2}\right)\right) \tag{27}$$

Now, let us substitute this into equation (26):

$$P(2, r) = 4\left(I(r) + \sum_{i=1}^{\lfloor r \rfloor} \left(I\left(\sqrt{r^2 - i^2}\right)\right) - I(r)\right) \tag{28}$$

The two $I(r)$'s cancel out and we obtain a simplified equation for P:

$$P(2,r) = 4\sum_{i=1}^{\lfloor r \rfloor}\left(I\left(\sqrt{r^2-i^2}\right)\right) = 4\sum_{i=1}^{\lfloor r \rfloor}\left(\left\lfloor \frac{1}{2}\left(\cos\left(2\pi\sqrt{r^2-i^2}\right)+1\right)\right\rfloor\right) \tag{29}$$

To find the number of integer lattice points in the closed disk, we need to take the summation of P over all r. This can be simplified, however, by taking the summation of y over all r discretely. To do so, we sum the number of $x \in \mathbb{Z}$ at each y over all r, using equation (20):

$$\sum_{i=0}^{\lfloor r \rfloor}\left\lfloor \sqrt{r^2-i^2}\right\rfloor \tag{30}$$

The floor function is included because it is not guaranteed that y is always an integer. We can also exclude $I(x)$ since we know x must be an integer. Now, we multiply this by 4 (due to the four quadrants) and add 1 for the origin point to obtain an exact explicit solution to the Gauss Circle Problem:

$$N(2,r) = 4\sum_{i=0}^{\lfloor r \rfloor}\left(\left\lfloor \sqrt{r^2-i^2}\right\rfloor\right)+1 \tag{31}$$

To determine the number of integer lattice points in the open disk, we subtract P from N:

$$O(2,r) = N(2,r) - P(2,r) = 4\left(\sum_{i=0}^{\lfloor r \rfloor}\left\lfloor \sqrt{r^2-i^2}\right\rfloor - \sum_{i=1}^{\lfloor r \rfloor}\left(I\left(\sqrt{r^2-i^2}\right)\right)\right)+1 \tag{32}$$

We can combine the two summations if we first separate the second term using equation (27):

$$\sum_{i=1}^{\lfloor r \rfloor}\left(I\left(\sqrt{r^2-i^2}\right)\right) = \sum_{i=0}^{\lfloor r \rfloor}\left(I\left(\sqrt{r^2-i^2}\right)\right) - I(r) \tag{33}$$

Substituting this into equation (32), we obtain a simplified equation for O:

$$
\begin{aligned}
O(2,r) &= 4\left(\sum_{i=0}^{\lfloor r\rfloor}\left\lfloor\sqrt{r^2-i^2}\right\rfloor-\sum_{i=0}^{\lfloor r\rfloor}\left(I\left(\sqrt{r^2-i^2}\right)\right)+I(r)\right)+1\\
&= 4\left(\sum_{i=0}^{\lfloor r\rfloor}\left(\left\lfloor\sqrt{r^2-i^2}\right\rfloor-I\left(\sqrt{r^2-i^2}\right)\right)+I(r)\right)+1\\
&= 4\left(\sum_{i=0}^{\lfloor r\rfloor}\left(\left\lfloor\sqrt{r^2-i^2}\right\rfloor-\left\lfloor\frac{1}{2}\left(\cos\left(2\pi\sqrt{r^2-i^2}\right)+1\right)\right\rfloor\right)+\left\lfloor\frac{1}{2}\left(\cos\left(2\pi r\right)+1\right)\right\rfloor\right)+1
\end{aligned}
\tag{34}
$$

Now, let us look at n-balls using concepts from the disk. The equation for an n-sphere is:

$$\sum_{j=1}^{n} {i_j}^2 = r^2 \tag{35}$$

where i is a coordinate. Since we wish to have only integer coordinates, then we include the Integer Test function:

$$\sum_{j=1}^{n} I\left(i_j\right) {i_j}^2 = r^2 \tag{36}$$

Let us separate i_n from the rest:

$$I\left(i_n\right) {i_n}^2 + \sum_{j=1}^{n-1} I\left(i_j\right) {i_j}^2 = r^2 \tag{37}$$

We can now rearrange this so that i_n is on the left and in terms of i_j:

$$I\left(i_n\right) i_n = \sqrt{r^2 - \sum_{j=1}^{n-1} I\left(i_j\right) {i_j}^2} \tag{38}$$

Due to the nature of the Integer Test function, $I\left(i_n\right)$ can be excluded:

$$i_n = \sqrt{r^2 - \sum_{j=1}^{n-1} I\left(i_j\right) {i_j}^2} \tag{39}$$

We now have an equation for $I\left(i_n\right)$:

$$I\left(i_n\right) = I\left(\sqrt{r^2 - \sum_{j=1}^{n-1} I\left(i_j\right) {i_j}^2}\right) \tag{40}$$

This is our counter function for an n-sphere with radius r. To calculate the number of integer lattice points on the n-sphere, we include a summation:

$$\sum I\left(\sqrt{r^2-\sum_{j=1}^{n-1} I\left(i_j\right) {i_j}^2}\right) \tag{41}$$

We know due to the discreteness of i_j, that each i_j must be an integer. Therefore, we can exclude the $I\left(i_j\right)$:

$$\sum I\left(\sqrt{r^2-\sum_{j=1}^{n-1} {i_j}^2}\right) \tag{42}$$

Now, we need to determine the lower and upper bounds. From equation (39), we can see that the bounds are dependent on the indices of the summation under the square root. Therefore, the outside summation we have in expression (42) is actually a series of summations, each with different bounds. Utilizing our convention for the circle, we set the lower bound of each summation to 0. The upper bounds, however, depend on the indices of the other summations, yet they follow the pattern from equation (39). To clearly state each of these summations and each of their bounds, we now define a set $\mathbb{I}$:

$$\mathbb{I}=\left\{\left(i_1, \ldots, i_{n-1}\right) \;\middle|\; 0 \leq i_k \leq\left\lfloor\sqrt{r^2-\sum_{j=k+1}^{n-1} {i_j}^2}\right\rfloor,\ 1 \leq k \leq n-1\right\} \tag{43}$$

The floor function is included because it is not guaranteed that the upper bound is always an integer. We take note that when $k=n-1$, the lower bound of the summation under the square root is n while its upper bound is $n-1$. For this case, we simply eliminate the summation (i.e. set it equal to 0). We can now apply this set to expression (42):

$$\sum_{\left(i_1, \ldots, i_{n-1}\right) \in \mathbb{I}} I\left(\sqrt{r^2-\sum_{j=k}^{n-1} {i_j}^2}\right) \tag{44}$$

Once again, there is double counting per half-axis. Thus, we need to subtract the point at the head of $\vec{v}$ if it is an integer:

$$\sum_{\left(i_1, \ldots, i_{n-1}\right) \in \mathbb{I}}\left(I\left(\sqrt{r^2-\sum_{j=k}^{n-1} {i_j}^2}\right)\right)-I(r) \tag{45}$$

Since we divide our space into half-axes and each axis corresponds to a dimension, we multiply expression (45) by $2n$ to obtain the number of integer lattice points on an n-sphere:

$$P(n, r)=2 n\left(\sum_{\left(i_1, \ldots, i_{n-1}\right) \in \mathbb{I}}\left(I\left(\sqrt{r^2-\sum_{j=k}^{n-1} {i_j}^2}\right)\right)-I(r)\right) \tag{46}$$

We can separate the first term from the case when $i=0$:

$$\sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) = I(r) + \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{J}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) \tag{47}$$

The new set $\mathbb{J}$ is defined as:

$$\mathbb{J} = \left\{ (i_1, \ldots, i_{n-1}) \;\middle|\; 1 \le i_k \le \left\lfloor \sqrt{r^2 - \sum_{j=k+1}^{n-1} {i_j}^2} \right\rfloor,\ 1 \le k \le n-1 \right\} \tag{48}$$

Now, let us substitute equation (47) into equation (46):

$$P(n,r) = 2n \left(I(r) + \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{J}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) - I(r) \right) \tag{49}$$

The two $I(r)$'s once again cancel out and we obtain a simplified equation for P:

$$\begin{aligned} P(n,r) &= 2n \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{J}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) \\ &= 2n \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{J}} \left(\left\lfloor \frac{1}{2} \left(\cos\left(2\pi \sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right) + 1 \right) \right\rfloor \right) \end{aligned} \tag{50}$$

To find the number of integer lattice points in an n-ball, we need to take the summation of P over all r. However, we can take the summation of i_n over all r discretely. In other words, we sum the number of $i_j \in \mathbb{Z}$ for $1 \le j \le n-1$ at each i_n over all r, using equation (39):

$$\sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left\lfloor \sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2} \right\rfloor \tag{51}$$

Recall the floor function is included because it is not guaranteed that i_n is always an integer. We also exclude $I(i_j)$ since we know i_j must be an integer. Now, we multiply this by $2n$ (due to the division of half-axes and each axis corresponds to a dimension) and add 1 for the origin point to obtain:

$$N(n,r) = 2n \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left(\left\lfloor \sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2} \right\rfloor \right) + 1 \tag{52}$$

To determine the number of integer lattice points in the open n-ball, we subtract P from N:

$$
\begin{aligned}
&O(n,r) = N(n,r) - P(n,r) \\
&= 2n\left(\sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left\lfloor\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right\rfloor - \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{J}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right)\right) + 1
\end{aligned} \tag{53}
$$

We combine the two summations if we first separate the second term using equation (47):

$$
\sum_{(i_1,\ldots,i_{n-1})\in\mathbb{J}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) = \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) - I(r) \tag{54}
$$

Substituting this into equation (53), we obtain a simplified equation for O:

$$
\begin{aligned}
&O(n,r) \\
&= 2n\left(\sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left\lfloor\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right\rfloor - \sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left(I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) + I(r)\right) + 1 \\
&= 2n\left(\sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left(\left\lfloor\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right\rfloor - I\left(\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right)\right) + I(r)\right) + 1 \\
&= 2n\left(\sum_{(i_1,\ldots,i_{n-1})\in\mathbb{I}} \left(\left\lfloor\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right\rfloor - \left\lfloor\frac{1}{2}\left(\cos\left(2\pi\sqrt{r^2 - \sum_{j=k}^{n-1} {i_j}^2}\right) + 1\right)\right\rfloor\right)\right. \\
&\qquad \left. + \left\lfloor\frac{1}{2}\left(\cos(2\pi r) + 1\right)\right\rfloor\right) + 1
\end{aligned} \tag{55}
$$

We have derived an explicit algorithm for calculating the exact number of integer lattice points on an n-sphere, in a closed n-ball, and in an open n-ball.

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