Computing the Fibonacci Sequence From a Single Row of Pascal's Triangle

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Abstract

This paper will derive an algorithm for computing the Fibonacci sequence from a single row of Pascal's triangle using a proven closed formula for the Fibonacci sequence.

Keywords

Fibonacci, sequence, single row, Pascal, triangle, algorithm

Introduction

The Fibonacci sequence is a sequence of numbers where the value that correlates with a particular index is equal to the sum of the values in the previous two indices and it is understood that the first two indices have the value of 1. A recursive formula is:

$$F_n = F_{n-2} + F_{n-1} \tag{1}$$

where n is the index in the Fibonacci sequence.

We will prove the claim that this summation is equal to a proven closed formula for the Fibonacci sequence and how it can be computed from a single row of Pascal's triangle:

$$F_n = \frac{1}{2^{n-1}} \sum_{i=1,odd}^{n} \binom{n}{i} \cdot 5^{\frac{i-1}{2}} \tag{2}$$

Proof

Using a stochastic matrix, we can determine the closed formula for the Fibonacci sequence[1]:

$$F_n = \frac{1}{\sqrt{5}} \left(\varphi^n - \phi^n \right) \tag{3}$$

where φ and ϕ are the two values for the golden ratio which satisfy the equation:

$$\varphi^2 - \varphi - 1 = 0 \tag{4}$$

which are:

$$\varphi = \frac{1+\sqrt{5}}{2} \tag{5}$$

$$\phi = \frac{1-\sqrt{5}}{2} \tag{6}$$

First, we notice that each of these values can be split into a sum or difference of two fractions:

$$\varphi = \frac{1}{2} + \frac{\sqrt{5}}{2} \tag{7}$$

$$\phi = \frac{1}{2} - \frac{\sqrt{5}}{2} \tag{8}$$

Second, we notice that each is to the power of n. The binomial theorem states:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^{n-i} y^i \tag{9}$$

$$(x-y)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot (-1)^i \, x^{n-i} y^i \tag{10}$$

Formula (3) includes a difference of the two golden ratio values, each with a power of n. This is equivalent to the difference of formulas (9) and (10):

$$(x+y)^n - (x-y)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^{n-i} y^i - \sum_{i=0}^{n} \binom{n}{i} \cdot (-1)^i \, x^{n-i} y^i \tag{11}$$

The summations in formula (11) are exactly the same except that for all the even indices, the terms are additive opposites. This means that the terms for all the even indices cancel each other out and the terms for odd indices double:

$$(x+y)^n - (x-y)^n = 2 \sum_{i=1,odd}^{n} \binom{n}{i} \cdot x^{n-i} y^i \tag{12}$$

Let us allow:

$$x = \frac{1}{2} \tag{13}$$

$$y = \frac{\sqrt{5}}{2} \tag{14}$$

If we substitute x and y into formula (12), we obtain:

$$\varphi^n - \phi^n = 2 \sum_{i=1,odd}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{n-i} \left(\frac{\sqrt{5}}{2}\right)^{i} \tag{15}$$

We can expand and simplify:

$$\varphi^n - \phi^n = 2 \sum_{i=1,odd}^{n} \binom{n}{i} \left(\frac{\sqrt{5}^i}{2^{n-i} \cdot 2^i}\right) = 2 \sum_{i=1,odd}^{n} \binom{n}{i} \left(\frac{\sqrt{5}^i}{2^n}\right) \tag{16}$$

Since the denominator does not depend on i, it can be moved outside the summation:

$$\varphi^n - \phi^n = \frac{2}{2^n} \sum_{i=1,odd}^{n} \binom{n}{i} \cdot \sqrt{5}^i = \frac{1}{2^{n-1}} \sum_{i=1,odd}^{n} \binom{n}{i} \cdot \sqrt{5}^i \tag{17}$$

Formula (3) includes a division of a square root of 5. Thus, we must do this in formula (17):

$$F_n = \frac{1}{2^{n-1} \cdot \sqrt{5}} \sum_{i=1,odd}^{n} \binom{n}{i} \cdot \sqrt{5}^i = \frac{1}{2^{n-1}} \sum_{i=1,odd}^{n} \binom{n}{i} \cdot \sqrt{5}^{i-1} \tag{18}$$

We can then change the square root to a division of 2 in the exponent:

$$F_n = \frac{1}{2^{n-1}} \sum_{i=1,odd}^{n} \binom{n}{i} \cdot 5^{\frac{i-1}{2}} \tag{19}$$

Since this formula matches formula (2), our claim has been proven. Pascal's triangle is a collection of numbers with rows corresponding to the binomial coefficient included in formula (9). The top number is the index of the row while the bottom number is the index of the column of the triangle. As we see in formula (2)/(19), the top number remains n, and therefore, the Fibonacci sequence can be computed from a single row of Pascal's triangle.

Reference

1. Notes on Linear Recurrence Sequences, Section 2.2
(http://people.math.gatech.edu/~ecroot/recurrence_notes2.pdf)