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Perfect Powers: The Jean Set and the Jean Function

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The Jean set, denoted $\mathbb{J}$, is the set of all perfect powers that are greater than one. That is the set of all values that can be represented by a base and exponent that are natural numbers greater than one. The set can be defined as the following:

$$\mathbb{J} = \{n^k : n, k \in \mathbb{N}, n, k > 1\}$$

I claim that that the cardinality of the set containing the intersection of $\mathbb{J}$ and the interval of $\mathbb{N}$ from two to *y* can be found using the Jean function, denoted *J(y)*:

$$J(y) = |\mathbb{J} \cap [2, y]| = \sum_{j=2}^{\lfloor\sqrt{y}\rfloor} \sum_{i=1}^{\lfloor \log_j \sqrt{y} \rfloor} \left\{[\sigma(i)]\left(\left\lfloor \frac{\log_j y}{i} \right\rfloor - 1\right)\right\}$$

$$\sigma(i) = \begin{cases} 1 \; if \; i = 1 \\ -1 \; if \; i > 1 \end{cases}$$

Alternatively, *J(y)* can be expressed in the following form:

$$J(y) = \sum_{j=2}^{\infty} \sum_{i=1}^{\infty} \left\{[\sigma(i)]\left(\left\lfloor \frac{\log_j y}{i} \right\rfloor - 1\right)\right\}$$

Since the definition of any element within the Jean set is simply n^k, where both *n* and *k* are natural numbers greater than one, we can state that the Jean set contains all powers of a particular *n*, except for when *k* = 1. This can be denoted as $\mathbb{J}_n$:

$$\mathbb{J}_n = \{n^2, n^3, n^4, n^5, n^6 \dots\}$$

Another way of stating this is that any $\mathbb{J}_n$ is a subset of the Jean set:

$$\mathbb{J}_n \subset \mathbb{J}$$

Therefore, we can conclude that the union of all the subsets $\mathbb{J}_n$ where *n* is greater than one are not only a subset of $\mathbb{J}$, but also equal to the set of $\mathbb{J}$:

$$\bigcup_{n=2}^{\infty} \mathbb{J}_n \subseteq \mathbb{J}$$

The cardinality of any $\mathbb{J}_n^{(y)}$ is the cardinality of the set containing the intersection of $\mathbb{J}$ and the interval of $\mathbb{N}$ from two to y:

$$\left|\mathbb{J}_n{}^{(y)}\right| = |\mathbb{J}_n \cap [2, y]|$$

If we added all the cardinalities of $\mathbb{J}_n$ together, we would obtain an upper bound cardinality of the set containing the intersection of $\mathbb{J}$ and the interval of $\mathbb{N}$ from two to y:

$$|\mathbb{J} \cap [2, y]| \leq \sum_{j=2}^{\infty} \left|\mathbb{J}_j{}^{(y)}\right|$$

This summation may be greater than the actual value because some elements will be the same in different subsets (*e.g.* $2^4 = 4^2$). We need to subtract the number of perfect powers that are duplicates so that they are only counted once.

We can state that a subset $\mathbb{J}_n$ where n is now a perfect power itself that can be expressed as a^b and both a and b are natural numbers, is a subset of $\mathbb{J}_a$ because the intersection of the two subsets would be $\mathbb{J}_a$:

$$\mathbb{J}_n \cap \mathbb{J}_a = \mathbb{J}_a$$
$$\mathbb{J}_n \subset \mathbb{J}_a$$

When n can be expressed as a^b, then $\mathbb{J}_{a^b}$ contains only redundant elements, which need to be removed from the final count. If we added all the cardinalities of $\mathbb{J}_{a^b}$ together in this equation, we would obtain the cardinality of the set containing the intersection of the union of all subsets $\mathbb{J}_{a^b}$ and the interval of $\mathbb{N}$ from two to y:

$$\left|\bigcup_{a=2}^{\infty} \bigcup_{b=2}^{\infty} \mathbb{J}_{a^b} \cap [2, y]\right| = \sum_{j=2}^{\infty} \sum_{i=2}^{\infty} \left|\mathbb{J}_{j^i}{}^{(y)}\right|$$

We can now subtract this from our first summation to obtain the actual cardinality of the set containing the intersection of $\mathbb{J}$ and the interval of $\mathbb{N}$ from two to y, and compress the function:

$$|\mathbb{J} \cap [2, y]| = \sum_{j=2}^{\infty} \left|\mathbb{J}_j{}^{(y)}\right| - \sum_{j=2}^{\infty} \sum_{i=2}^{\infty} \left|\mathbb{J}_{j^i}{}^{(y)}\right| = \sum_{j=2}^{\infty} \sum_{i=1}^{1} \left|\mathbb{J}_{j^i}{}^{(y)}\right| - \sum_{j=2}^{\infty} \sum_{i=2}^{\infty} \left|\mathbb{J}_{j^i}{}^{(y)}\right|$$
$$= \sum_{j=2}^{\infty} \sum_{i=1}^{\infty} \left[\sigma(i) \left|\mathbb{J}_{j^i}{}^{(y)}\right|\right]$$

As we can see, the sum in the first piece is positive, but the sum in the second piece is negative when we subtract. We also know that in the first piece, *i* only equals one, while in the second piece, *i* is always greater than one. From this information, I call the new subfunction denoted *σ(i)*:

$$\sigma(i) = \begin{cases} 1 \; if \; i = 1 \\ -1 \; if \; i > 1 \end{cases}$$

The cardinality of $\mathbb{J}_{j^i}(y)$ is simple to determine through the use of logarithms:

$$\left|\mathbb{J}_{j^i}(y)\right| = \left\lfloor \frac{\log_j y}{i} \right\rfloor - 1$$

We can now substitute this to obtain the alternate Jean function:

$$J(y) = \sum_{j=2}^{\infty} \sum_{i=1}^{\infty} \left\{ [\sigma(i)] \left(\left\lfloor \frac{\log_j y}{i} \right\rfloor - 1 \right) \right\}$$

Of course we could leave this as is, but the upper bounds do have a limit to when you evaluate the function after a certain *i* and *j* term, we would continuously be adding zero.

Let's begin with the outer summation. The *j* term in this summation represents the base, or the subset $\mathbb{J}_j$. The upper bound then needs to be the greatest base that when raised to the smallest power, it does not exceed the value of *y*. As defined by the Jean set, the smallest exponent any base can have to be included in the Jean set is two. Therefore, the upper bound for the outer summation will be the greatest base that when squared is equal to or no greater than *y*:

$$j^2 \leq y$$
$$j \leq \sqrt{y}$$
$$j \leq \lfloor \sqrt{y} \rfloor$$

The *i* term in the inner summation represents the exponent to any base that may be a perfect power. The upper bound then needs to be the greatest exponent for each base so that when the base is raised to that power, it does not exceed the value of the upper bound for the outer summation. The upper bound for the inner summation can be found as follows:

$$\left(j^i\right)^2 \leq y$$
$$j^i \leq \sqrt{y}$$
$$i \leq \log_j \sqrt{y}$$
$$i \leq \lfloor \log_j \sqrt{y} \rfloor$$

We can substitute these two upper bounds into the alternate Jean function to obtain an equivalent Jean function:

$$J(y) = |\mathbb{J} \cap [2, y]| = \sum_{j=2}^{\lfloor\sqrt{y}\rfloor} \sum_{i=1}^{\lfloor\log_j \sqrt{y}\rfloor} \left\{[\sigma(i)]\left(\left\lfloor\frac{\log_j y}{i}\right\rfloor - 1\right)\right\}$$

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